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On spinor wave equations containing reflection operators

By H. A. S. ERIKSSON

The spinor formalism as developed in our earlier works (1943, 1953) has proved to be very suitable when investigating the various possibilities for constructing wave equations containing reflection operators. In spite of the fact that such equations do not fit well in the usual canonical scheme, it may nevertheless be of interest to see what physical results they would give. In the present preliminary note, we will apply such a wave equation, obtained by transformation considerations which will not be described here, to an electron in a Coulomb field.

We have chosen the spinor equation

$$\left.\begin{aligned}(D_1 + i D_2) u_1 - (D_3 - D_4) u_2 + \varkappa P_2 u_1 = 0,\\ (D_1 - i D_2) u_2 + (D_3 + D_4) u_1 - \varkappa P_2 u_2 = 0,\end{aligned}\right\}$$

where the reflection operator P_2 is defined by

$$P_2 u_r (x_1, x_2, x_3, x_4) = u_r (x_1, -x_2, x_3, x_4), \qquad (r = 1, 2)$$

and where

$$D_\mu = \partial_\mu - i U_\mu = \partial / \partial x^\mu - i U_\mu, \qquad x^4 = -x_4 = c t, \qquad \varkappa = m c / (h / 2 \pi).$$

In our application we have

$$U_1 = U_2 = U_3 = 0, \qquad U_4 = \beta / r,$$

where β is the fine structure constant.

For a stationary state we may put

$$u_1 = e^{-i \varepsilon x_4} \varphi_1, \qquad u_2 = e^{-i \varepsilon x_4} \varphi_2,$$

where φ_1 and φ_2 are functions of x_1, x_2, x_3

$$\left.\begin{aligned}(\partial_1 + i \partial_2) \varphi_1 - \partial_3 \varphi_2 + i (\varepsilon + \beta / r) \varphi_2 + \varkappa P_2 \varphi_1 = 0,\\ (\partial_1 - i \partial_2) \varphi_2 + \partial_3 \varphi_1 + i (\varepsilon + \beta / r) \varphi_1 - \varkappa P_2 \varphi_2 = 0.\end{aligned}\right\}$$

New functions ψ_1 and ψ_2 are defined by the relations

$$\psi_1 = \varphi_1 + i\,\bar{\varphi}_2, \qquad \psi_2 = -i\,\bar{\varphi}_1 + \varphi_2,$$

where $\bar{\varphi}_r$ means the function complex conjugated to φ_r.

$$\left.\begin{aligned} &(\partial_1 + i\,\partial_2)\,\psi_1 - \partial_3\,\psi_2 + i\,(\varepsilon + \beta/r)\,\psi_2 - i\,\varkappa\,P_2\,\bar{\psi}_2 = 0, \\ &(\partial_1 - i\,\partial_2)\,\psi_2 + \partial_3\,\psi_1 + i\,(\varepsilon + \beta/r)\,\psi_1 - i\,\varkappa\,P_2\,\bar{\psi}_1 = 0. \end{aligned}\right\}$$

Separation of the angles can be achieved by putting

$$\left.\begin{aligned} \psi_1 &= \sqrt{\frac{l-\mu}{2\,l+1}}\; Y_{l,\mu}\,g(r) + i\sqrt{\frac{l+\mu}{2\,l-1}}\; Y_{l-1,\mu}\,f(r), \\ \psi_2 &= \sqrt{\frac{l+\mu+1}{2\,l+1}}\; Y_{l,\mu+1}\,g(r) - i\sqrt{\frac{l-\mu-1}{2\,l-1}}\; Y_{l-1,\mu+1}\,f(r); \end{aligned}\right\}$$

$$\left.\begin{aligned} &\frac{d\,g}{d\,r} + \frac{l+1}{r}\,g - \left(\varepsilon + \frac{\beta}{r} + \varkappa\right) f = 0, \\ &\frac{d\,f}{d\,r} - \frac{l-1}{r}\,f + \left(\varepsilon + \frac{\beta}{r} - \varkappa\right) g = 0. \end{aligned}\right\}$$

Since these equations are the same as those occurring in the ordinary Dirac theory, they will give the well-known formula for the energy. (It should be mentioned that the four-dimensional divergence of the current density vector does not vanish but, since it turns out to be an odd function of x_2, its space integral vanishes. Therefore, the normalization of the wave function is independent of time.)

The equations above are covariant with respect to restricted (continuous) Lorentz transformations. A system of equations covariant also with respect to space reflections can be obtained by introducing another pair of equations in addition to those treated here.

SUMMARY

An example is given of a system of two first order wave equations containing a reflection operator. Applied to an electron in a Coulomb field, the ordinary fine structure formula is obtained.

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